Student Development in the Understanding of Proof

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Students commonly experience frustration as they make the transition from calculus to abstract mathematics. This shift from algorithmic-type computations to structured thinking (as in mathematical proofs) leaves many feeling lost and confused. In an effort to aid in this transition, many colleges offer a course specifically designed to teach mathematical proof techniques. Such a course typically provides the students with a study of formal logic leading to proof techniques.

While the authors maintain that such a course can be extremely useful to student development, there are many questions that come to mind as we try to structure a transitional course. Does the course really help students in their understanding of proof? Is it really a *transition* course (or is a course in logic or some other hodgepodge of topics)? When we attempt to teach proofs in such a course, we are in a sense trying to teach in an algorithmic fashion something that takes a great deal of creativity and understanding. Is it possible for students to make the transition from algorithmic type thinking in an environment where they are introduced to proofs in an algorithmic fashion? Is it reasonable for students to develop a true ability to prove in such a one semester course? Do we have a real understanding of how to teach the proof process? These questions provided the motivation for this project.

Similar to many other institutions of higher learning, Millsaps College offers a course specifically designed to teach students how to read, understand, and construct mathematical proofs. In this course, approximately the first third of the semester is dedicated to logic, the second third to teaching some standard proof techniques (direct, contradiction, contrapositive, induction), and the last third to introducing the students to various areas of mathematics (set theory, relations, functions, cardinality, etc.) and in so doing, to a variety of mathematical definitions, axioms, and to mathematical proofs in a variety of settings. Our project involved students from two sections of this course.

In an effort to better understand how students develop in their understanding of proof, the authors decided to try to gain a window into student perceptions of proof as they were enrolled in this bridge course. To this end we (among other things) videotaped students as they worked together in groups. The first of these videotaping sessions occurred after the students finished the logic section and the second (which will be the focus of this paper) as they were finishing the section on proofs. We set up the experiment as follows: Students from two sections of our bridge course (same text, different instructors) were asked to participate in our research project. For this project, the students were to form groups of size 2 or 3 and to sign up for a 30 minute session where their group would be given a set of statements to prove or disprove. The students were informed that we were not looking for whether they were 'right or wrong' in their work, but rather what thought processes they used, how they communicated ideas, and how they worked as a group to approach proving mathematical statements.

The students were assured that their instructor would not have access to the tapes until after semester grades were due so that this material could in no way be used in their course grade evaluation. Of a total of 19 students, 14 students chose to participate in our project. These students self-selected into six groups; four groups of size 2 and two groups of size 3.

The tapings occurred approximately at mid-term, the point in the semester when students had been introduced to proof techniques and some basic definitions (even, odd, etc.). Hence, we believe that these students were at a very crucial junction. They were just beginning to gain an understanding of what it meant to prove a statement and how to proceed to do so.

The authors wanted to see how the students would approach trying to prove, or disprove, six different types of statements: prove/disprove a true statement, prove/disprove a false statement, a statement on divisibility, a statement which would necessitate looking at indirect methods of proof, an if and only if statement, and an inductive statement. We designed a set of statements to represent these types, but due to the 30 minute time constraint did not have all groups work with all statements. Instead we divided the statements into two groups of four (with some overlap). Hence, each group was given four statements to consider (see Appendix).

The first statement for all groups was one which they were specifically asked to prove or disprove. We set up the sessions so that half of the groups would get a true statement and so would need to produce a proof, while the other half would get a false statement and so would need to produce a counterexample.

The second problem was set up so that half of the groups were given a statement on divisibility (direct proof would be easy) and the other half were given a statement on the real numbers (which would be difficult for students to prove if they didn't use an indirect method).

The third and fourth statements were the same for all groups. The third being an 'if and only if' statement and the fourth, a proof by induction.

Observations on statement 1 (A&B): Groups did not even consider that the statement might be false. They immediately attempted to prove the statement and all groups thought they did just this. In fact, only one of the six groups constructed a valid argument for statement 1. Both of these statements involved even and odd integers and, even at the expense of showing the converse, all groups started with the definition of an even (odd) integer.

Observations on statement 2A: Student interaction on this statement provided us with a source of great discussion. The students recognized a complexity in this statement and when an immediate idea did not come to mind, they resorted to logic. Each group attempted to transcribe the statement into ‘p and q implies r’ form, and discussed what they could assume if they proceeded by a contradiction or contrapositive argument. Through this discussion, many students demonstrated that they did not understand the difference in these two approaches. All groups had a great deal of trouble developing a contradiction argument.

Students struggled on this problem but were tenacious in their attack. Even when one group was given permission to stop working on this problem (due to time), they proceeded to keep working on it. All groups thought they proved the statement but in fact, no group was successful.

Of particular interest was a group who thought they were successful as they provided an example of where the negation of the statement failed. Thus they believed that by showing an example of where the negation was false they had proven that the original statement was in fact true. In actuality they have given an example of where the theorem is true.

Observations on statement 2B: As students worked on this statement, they either started in a rather mechanical fashion (rote memory) or that they had no idea of where to start.

Observations on statement 3: While not all groups were successful in proving statement 3, the only mistake made involved the misconception of claiming that if $a < b$ then $b = a+1$ (this misconception occurred within 2 of the groups). The authors did note that the students did not seem to think about trying to prove either direction of this statement via an indirect method. The students appeared to be confident that as the statement was ‘if and only if’, they were to prove both directions, and each direction should be proven by a direct method.

Observations on statement 4: The students we observed experienced the greatest level of success with proofs by induction. Every group immediately identified the statement as an inductive statement and all groups were successful in constructing a valid proof. Common was the statement “Oh, this is one of those induction problems.” They very systematically approached proving this statement. Common was a remark such as “Okay, we have to show those 3 things and then we’re done.” It was also interesting to note that while all groups verbally noted that they could not assume the statement was true for k+1, many of them invariably wrote this statement into their proof (that is they included within the body of their proof the induction equation for k+1).

Conclusions:
1) Students who prove the converse may do so as they are trying to ‘latch on’ to what they know. As noted in our observations, several groups proved the converse of statement 1. These groups invariably began by writing the definition of an even (odd) integer and working with this definition. The authors even observed one group who proved the converse and then, as they reviewed their argument, realized their mistake and started over, ultimately constructing a valid proof. Proving the converse is a common mistake in an early proofs course and research has been done on why students make this error. Seldon and Seldon [3] proposed that this may be due to the difference between everyday language and mathematical language (specifically that people often say “if” in everyday language when they mean “if and only if”). While Seldon and Seldon present a good argument, it was interesting to the authors that this error only occurred as our subjects worked on statement 1. This statement was the only one whose conclusion contained something

for which the students possessed a well known definition. We conjecture that the issue was not that the students did not understand the 'if-then' format, but rather that they wanted to begin with what they knew. If what they knew happened to be in the 'then' portion of the statement, they often (in the heat of jumping into a solution) started with that information.

2) Beginning proof students do not see the need to have an understanding of the statement before (or after) they attempt to prove it. This was especially evident in the way students approached working with statements 1 and 2B. Even on statement 1 where the directions specifically stated prove/disprove, students never once tried an example to see if they believed the statement. Similarly on statement 2B, when students were stuck on how to proceed, no one ever tried an example to help them understand the statement or to help them formulate a definition of divisibility.

3) Students see proving theorems as problem and solution rather than providing insight to structure. Even though they understood that the process was the answer and that each line of their proof was significant, these students were still in a very algorithmic approach to proofs. They saw proofs as problems to be solved (similar to problems to which they had been exposed in prior mathematics classes). This problem-solution approach was observed as students discussed each of the statements. In no case did students indicate that the content of the statement was of any relevance. They did not appear to understand the role of proof as a way of gaining insight to the statement. Hanna [1] differentiates between proofs that prove, and proofs that explain. The students we observed gave no indication of understanding the role of proof as a means of explanation. Rather they seemed to understand proofs as a new type of mathematical problem for which an algorithmic proof was a solution.

4) Students place a high importance on devising a plan. While students spent virtually no time discussing what the statement meant mathematically, they did spend a great deal of time talking about logic and what approach they should take in attempting to prove the statement As observed on statement 2B, groups who could not develop a means of attack were paralyzed. The authors would have hoped that when students could not identify a means for starting their proof, that they would try some examples and attempt to gain insight into the meaning of the statement as well as some ideas as to why it might be true. Polya noted that students approach problem solving by the following four step method: attempt to understand the problem, devise a plan, execute the plan, review the plan. The students we observed utilized this model, with the exception of the first step on each statement. That these students, who had never been exposed to Polya's method, followed this model so closely supports our inferences that the students were approaching theorem proving as problem solving and offers us an explanation of why they did not see the need to understand the statement in order to prove it.

5) Students understand that the study of logic provides techniques of verifying proofs, but again, they see proofs as algorithmic in nature and still have not come to the point where they understand the context of what they are doing. This was especially evident on statement 2A where, when students were unsuccessful in producing a direct argument, they immediately resorted to logic. Harel [2] introduced several different schemes which students used as they approached proving theorems. For statement 2B, the only proof for which students really needed to use an indirect method, students utilized a scheme Harel termed as the 'symbolic proof scheme'. That is, the

students converted the problem to symbolic form without trying to understand the context of the problem. Only after transcribing the statement to symbolic logic could they seem to make any sense of how to prove the statement by an indirect means. From their conversations it was evident that the students knew that a contrapositive or contradiction argument would be logically equivalent to a direct argument. While these students had a hard time constructing a contradiction argument and/or the contrapositive without resorting to symbolic logic, they understood how to use formal logic to help them construct these arguments.

6) Students have trouble understanding the logic behind a proof by contradiction. As noted above, when students sought an indirect method to proving statement 2A, they immediately resorted to the logic notation of p's and q's. Students appeared to be quite comfortable understanding the contrapostive argument and why that was a logical equivalent of the statement. However, when they were unable to prove the statement by this means, they spent a significant amount of time trying to understand how a contradiction would help them. Students seemed to understand that the contradiction involved the negation but they struggled in understanding how utilizing a false statement would allow them to make any conclusions about the original statement. Common was a statement along the lines of the following: "We know the statement is true so it's negation is false, and false implies anything is true, so how does that help us?" Also of interest was a group who realized that they needed prove the negation was false and reasoned that this could be done by producing an example of where the negation failed to be true. These students were remembering that they could disprove a statement by producing a counterexample but what they were actually doing was giving an example of where the original statement was true.

7) Students find 'if and only if' statements logically easy to understand. In contrast to statements 2A&B which either immobilized students or caused them a great deal of discussion on how to proceed, students were quick to attack statement 3. They knew that they must prove the statement 'both ways' and hence did not get confused (or even spend any time discussing) what they were, or were not, allowed to assume. Students noted that the statement was 'if and only if', and immediately (and algorithmically) picked half of the statement to assume, proved it, and then assumed the other half and proved the reverse. As mentioned in our observations, the authors believe that the students may have the misconception that all 'if and only if' statements must be proved by a direct method, even though these students were exposed to theorems in class where this was not the case.

8) Beginning proof students write proofs more for themselves than for others. These students appeared to see proofs more as a dialogue of their thought process than as an argument which was constructed to convince someone else of the validity of a theorem. This was evident from the fact that, while they understood perfectly well that they weren't allowed (in an induction proof) to assume the statement was true for k+1, almost all groups invariably wrote this into their proof. Their verbal comments made it clear to the authors that the students were not planning to use this statement but rather were thinking about what they needed to show. The students who wrote this equation in their proof never noticed that what they wrote literally meant that they were assuming what they were trying to prove. It was clear from their verbal expressions that these students were not assuming anything that they shouldn't.

Final Conclusion: Students are receptive to the idea that a proof is a means of demonstrating the validity of a theorem and they see how logic relates to proofs. What they don't see is how proof gives insight into meaning. This level of understanding has been noted by others. Skemp [5] defined relational understanding and instrumental understanding. Briefly stated, instrumental understanding is an understanding of 'how to' while relational understanding is an understanding of 'how things fit together'. At the time of videotaping these students were predominately at a level of instrumental understanding. A bridge course can be successful in the sense that students are given (and are receptive to) the tools which they will need in future, more abstract courses. However, this course (at least at mid-term) was unsuccessful in providing students with a greater understanding of what mathematics is all about. Some of this could be due to the emphasis which was placed on logic during the first part of the course. In fact, some present this as a compelling argument to remove logic from a bridge course. While the authors see the merits of this argument, we agree with Selden & Selden [4] in that we see more dangers in omitting the study of formal logic. In fact, one of the authors taught this same bridge course with an added unit on conjecture and a much lesser emphasis on logic. This resulted in a different problem. These students were much stronger in understanding the substance of theorems, but were very weak in their ability to prove theorems. Most of the students in this modified course felt that mathematics required a level of creativity which was out of their grasp. The class was more disheartened and, in general, saw mathematics as something very different from their previous experience. This left many students wondering if what they loved about mathematics was not to be found in any mathematics class beyond the calculus level. That is, they missed the problem solving that they enjoyed in lower level classes. Most of these students did not continue into a higher level mathematics course. This was disappointing in that, while these students would probably have never made it to be professional mathematicians, they would have been strong majors at the undergraduate level and would have been quite successful in the types of jobs for which undergraduate majors are often solicited.

What's Next: Now that we have seen some of these commonalities, we will teach the course again with an emphasis on proving to understand. Toward this end, we plan to add an emphasis on conjecture as well as a unit on reading proofs. Because this will add to the content which will need to be covered before introducing the students to writing proofs, we wonder if perhaps the better transition course model would be a two semester course. In such a setting, students could have more opportunities to see proofs in various areas of mathematics and to have the opportunity to see how reading proofs aids in the understanding of mathematical concepts. Alternatively, it has been suggested that a better model is to introduce proofs in smaller doses throughout the curriculum (beginning in calculus).

We will also continue to study some of the students we videotaped as they move into upper level mathematics classes and monitor their progress to see if we can determine where milestones in mathematical understanding occur. The authors are confident that these students will develop relational understanding but are not sure at what point this transition will actually occur.

Appendix:

1A. Prove or disprove the following: For any integers m and n, if $m \cdot n$ is even then m is even and n is even.

1B. Prove or disprove the following: For any integers m and n, if $m + n$ is odd then m is odd or n is odd.

2A. Prove the following: If x and y are positive real numbers and $x \neq y$, then $\frac{x}{y} + \frac{y}{x} > 2$.

2B. Prove the following: If n is divisible by 7 then $n^2 + 2n - 14$ is divisible by 7.

3. Prove the following: Let a and b be two distinct real numbers. Prove that a < b if and only if $\frac{a+b}{2} > a$.

4. Prove that for any natural number n, $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$.

References:

[1] Hanna, G. (1989). Proofs that Prove and Proofs that Explain. *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education*, Vol. 2, Paris, France, 45-51.

[2]: Harel, G. & Sowder, L. (1996). Classifying Processes of Proving. *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education*, Vol. 3, Valencia, Spain, 59-65.

[3]: Selden, A. & Selden, J. (1987). Errors and Misconceptions in College Level Theorem Proving. *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*, Vol. III, Cornell University, Ithaca, NY, 456-471.

[4]: Selden, A. & Selden, J. (1996). The Role of Logic in the Validation of Mathematical Proofs. The DIMACS Symposium on Teaching Logic and Reasoning, Rutgers University, 25-26 July 1996.

[5]: Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, Vol. 77, 20-26.