Group Dynamics in Learning to Prove Theorems

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We began this research project as an investigation into how students develop in their understanding of proof. The authors wanted to gain a window into student perceptions of proof, but along the way, began also to catch glimpses of group dynamics. The participants in this study were students enrolled in a bridge course designed to help students make the transition from calculus to higher mathematics. The course, like similar courses at many colleges and universities, attempts to provide students with a foundation of logic followed by instruction in proof techniques. The data used for the basis of this study were a series of videotaped sessions where students worked in groups to prove (or disprove) a series of mathematical statements. The purpose of this experiment was to help provide insights into student approaches to proving. Hence, the motivation for having the students work in groups was so that we could listen in as they discussed proving strategies with peers and hopefully gain insights into their thought processes.

Students from two sections of a bridge class were asked to participate. The sections had different instructors but utilized the same text. These students were asked to form groups of size 2 or 3, and to sign up for a 30 minute session where they would be asked, as a group, to prove or disprove a set of four statements. The students were assured that we were not looking for whether or not their work was 'correct', but rather what thought processes they used, how they communicated ideas with each other, and how they worked as a group to construct proofs. Students were also assured that their instructor would not view the tapes before semester grades were due and so their performance would not, in any way, affect their course grade. Of the 19 students enrolled in the two classes, 14 chose to participate in the project. The taped sessions occurred at approximately the time in the bridge course when students were just beginning to be successful in understanding how to prove elementary concepts.

The authors wanted to observe how students would approach the following types of statements: prove-disprove a true statement, prove-disprove a false statement, $p \rightarrow q$, $(p \wedge \sim q) \rightarrow r$, if and only if, and an inductive statement. We constructed six statements, one of each type, but due to the 30 minute time constraint, did not require all groups to prove all six statements. Instead, we separated the statements (with some overlap) into two sets of four as follows:

Set A:
1. Prove or disprove the following: For any integers m and n, if $m \cdot n$ is even then m is even and n is even.
2. Prove the following: If x and y are positive real numbers and $x \neq y, then \frac{x}{y} + \frac{y}{x} > 2.$
3. Prove the following: Let a and b be two distinct real numbers. Prove that a < b if and only if $\frac{a+b}{2} > a$.
4. Prove that for any natural number n, $1 + 2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$.

Set B:
1. Prove or disprove the following: For any integers m and n, if m+n is odd then m is odd or n is odd.
2. Prove the following: If n is divisible by 7 then $n^2 + 2n - 14$ is divisible by 7.
3. Prove the following: Let a and b be two distinct real numbers. Prove that a < b if and only if $\frac{a+b}{2} > a$.
4. Prove that for any natural number n, $1 + 2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$.

The video camera was set up so that the students were required to stand together in front of a section of chalkboard approximately five feet in length. They were given the set of statements and instructed to discuss their ideas for proving (or disproving) each statement. They were encouraged to write any preliminary work on the board. They were also instructed that when the group finalized their proof, they were to write the proof in final form on the board and to have one student turn to the camera and read through their proof. This final step was intended so that the students would have a real sense of closure and so they would be, as a group, ready to move on to the next statement.

The students self-selected into 6 groups. Groups 1 and 2 consisted each of two females, Group 3 of two males, and Group 4 of one female and one male. The remaining two groups had three members, Group 5 consisted of 3 males, and Group 6 of 3 females. Three of the groups were given the statements from Set A, and three were given the statements from Set B. For future reference, groups 1, 4, and 6 were given Set A and groups 2, 3, and 5, Set B.

As previously stated, the focus of this study was on student understanding of proof. However, as we reviewed the tapes and observed the students as they worked on proving these statements, we were struck by the group dynamics and how this affected theorem proving.

All the students, independent of the size of their group, the set of statements the group was assigned, or the gender structure of the group, showed extreme tenacity and persistence in working on the proofs. For example, Group 4 spent a great deal of time discussing the contrapositive and proof by contradiction as a means of proving statement 2. After some time, they still had not finalized a proof and were told that they could move on the next statement. In response, they expressed a desire to continue working on this statement and did so until they were satisfied with their results. Similarly, after several attempts, Group 5 was unable to prove statement 2. They also were instructed that they could proceed on to the next statement. In contrast to Group 4, this group did move on to statement 3. However, after finishing the fourth and final problem, they expressed a desire to go back and look at statement 2 again.

Many groups took far longer to formulate a proof that we expected. We looked on, unable to comment or give help, as groups struggled with a proof, pursued several incorrect paths, but were finally successful. This prompted the authors to wonder how often we preempt our students from making self discoveries. In several cases we were convinced that a group was destined for failure, but were proven wrong as they finally came around to a valid argument. Had the environment been different, and we been able to interact with the students, we most certainly would have offered input, potentially sending the incorrect message to these students that they could not have developed a correct solution independently.

All students also appeared genuinely interested in working on each statement until they reached complete agreement on a solution. Even though almost all sessions exceeded the scheduled 30 minute time slot, no student ever looked at a clock or inquired about the time. These students never gave the appearance of being concerned about time. It would be interesting and informative to know if these same students would have exhibited such interest and persistence had they been working alone. The authors seriously doubt that this would be the case.

Another characteristic of all groups was the high level of confidence and enthusiasm they displayed in their work. All students were engaged in the process and, whenever they completed a proof, the students appeared to be genuinely proud of their results and eager to move on to the next statement. Admittedly, there were cases when the groups displayed a true pride in a fundamentally flawed proof. For example, Group 1 proved the converse of statement 1 and truly believed that they had done an excellent job of proving the original statement. When their 'proof' was complete, they showed visible signs of enthusiasm. They smiled widely and 'jumped up and down' as they read their final product. However, not all the enthusiasm we observed was misdirected. There were times when we watched as ideas clicked. Each time a new idea was generated, the students displayed excitement and, when the group finally agreed they had completed their work, the students showed a true group pride in their results. One of the challenges of teaching is to be able to give students the constructive criticism they need without squelching their enthusiasm.

Group size had a definite impact on the group dynamics. In Groups 5 and 6 (the three member groups), there appeared to be two dominant members and a follower, but role reversals occurred within this structure. For example, in Group 6, two of the students began working to the exclusion of the third. They barely acknowledged the few comments that this student made. When they reached statement 3 (after being unsuccessful in finishing problem 2), the excluded student asserted himself in formulating the beginning of a proof. The two former leaders offered no help or suggestions. In fact, they showed physical signs that they fully intended to leave him to his own demise. They backed away from the board, crossed their arms across their chest, and exchanged a sneered glance behind his back as he worked on statement 3. The student tried several approaches to proving the statement and was unsuccessful but he displayed the persistence that was so common and continued to ponder what to try next. After a long pause, one of the other students offered a tentative suggestion: "Use k and k+1—something like that?" The student then substituted k and k+1 for a and b respectively in the inequality, but was not convinced that it was a useful assumption. However, a brief discussion satisfied all three of them, and the student proceeded, completing the proof. Unfortunately the assumption that a = k and b = k+1 was incorrect, but it was at this point that the group began to truly function like a team. While proving statement 4, all three students were writing, discussing, and contributing to the proof.

Group 6 also had a passive member, who quietly let the dominant pair take control. One of the pair did the writing and the other did most of the talking. The writer often asserted herself by discussing the proof strategy and making changes or improvements in the proof. As the group began working with statement 4, the inductive statement, the dominant pair made a mistake and the passive member began to challenge what the other two students were doing. They dismissed her arguments, but she demonstrated a great deal of confidence and was not willing to be brushed aside so easily. She requested paper and pencil, and proceeded to work the proof independently. When she finished she showed the others her work.

In contrast to this, the groups of two worked cooperatively without any signs of competition or a dominant member. The only exception was Group 3, both strong students. They worked separately at the board, stopping to look at what the other was doing and to discuss. They worked efficiently, completing their session in the shortest time of all the groups and with the best overall performance. There were no signs of competition or dominance, but it was not cooperative learning. Rather this illustrated a different group dynamic, one of collaborative and mutually supportive work. While most groups used each other to generate and formulate ideas, this group offered each other support and affirmation.

One surprising observation came when Group 6 devised a tool that they perceived worked well. This tool carried great weight and they continued to use it as they worked on the next statement. Specifically, in statement 2, they agreed that x and y positive real numbers with $x < y$, allowed them to conclude $y = x + 1$. When this technique worked (in the sense that they were able to complete their proof to their satisfaction), they used it again in a similar situation in statement 3. It was interesting that Group 5 also used this same misconception and, consequently, the authors plan to do a post-proof interview in which we will ask these students how they came to this conclusion. As the group accepted a method that they had constructed together, we were impressed by the social aspect of group work. Some studies have been done on the theory of social constructivism in mathematics, and indeed this raises some interesting questions: How much of what we consider mathematical object is social? How much of the acceptance of a mathematical object is social? How much did social factors affect the method that a group used to attack a problem and how much did it affect the solution?

This project demonstrated to us how powerful group work can be in a mathematics course. On one hand, without guidance, groups can perpetuate misconceptions and poor proving techniques. On the other hand, such groups provide a forum in which students can verbalize their thoughts and learn from each other. In the words of Selden and Selden, "the opportunity to 'talk mathematics' is thought to be beneficial not only for developing communication skills, but also for cognition". As a result of this study, the authors intend to make group proving sessions with feedback from the instructor a part of the bridge courses that we teach.