The Representations of GL(2,K) where K is a finite field

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1 Introduction

In this article, we are concerned with the irreducible complex representations of $GL(2, K)$ where K is a finite field with at least three elements. We will sketch the procedure given by Ilya Piatetski-Shapiro in [1] in constructing the (complex) irreducible representations of $GL(2, K)$. One-half of the irreducible representations of $GL(2, K)$ are the so-called cuspidal representations. If L is the quadratic extension of K, then the nondecomposable characters of L^* modulo a certain "conjugation" (to be defined later) correspond bijectively with the cuspidal representations of $GL(2, K)$ where $L^* = L - \{0\}$. We will conclude this article with an example wherein we describe the nondecomposable characters of $GF(49)^*$ where $GF(49)$ is the quadratic extension of the field $GF(7)$ with seven elements.

We begin by recalling a few preliminaries. Let V be a finite dimensional complex vector space and let $GL(V)$ be the group of linear isomorphisms of V. By a representation of $GL(2, K)$ we mean a homomorphism $\phi : GL(2, K) \to GL(V)$. A subspace $W \subseteq V$ is invariant if $\phi(g)W = W \ \ \forall g \in GL(2, K)$. If V and $\{0\}$ are the only invariant subspaces of V, then the representation is said to be an irreducible representation. The character χ_ϕ is the complex-valued function defined by $\chi_\phi(g) = \mathrm{Tr}(\phi(g))$, i.e., the trace of $\phi(g)$. The dimension of ϕ or χ_ϕ is the dimension of V. For abelian groups, every irreducible representation has dimension one.

In the next section, we briefly describe the irreducible representations of $GL(2, K)$.

2 Irreducible Representations

First, we list the number of conjugacy classes in $GL(2, K)$. This is needed since the number of conjugacy classes is the same as the number of irreducible representations over $\mathbb{C}$.

Given $m \in GL(2, K)$, the conjugacy class of m is the set $\{gmg^{-1} : g \in GL(2, K)\}$. We assume $|K| = q \geq 3$. We will choose a canonical element from each conjugacy class. If $\alpha, \beta \in K$ are non-zero and distinct, then the following matrices have distinct conjugacy classes and we list the number of conjugacy classes whose representatives can be expressed in such canonical form.

Conjugacy class representative	Number of conjugacy classes	Number of elements in conjugacy class
$c_1(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$q-1$	1
$c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$q-1$	q^2-1
$c_3(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$\frac{(q-1)(q-2)}{2}$ where $\alpha \neq \beta$	q^2+q

There are other conjugacy classes. Let L be a quadratic extension (this is unique up to isomorphism) of K. If the eigenvalues of m lie in L and not in K, then the characteristic polynomial of m in the indeterminate t can be written as $t^2-(\alpha+\overline{\alpha})t+\alpha\overline{\alpha}$ where $\alpha, \overline{\alpha} \in L-K$, $\alpha \neq \overline{\alpha}$. Note, $\overline{\alpha} = \alpha^q$. If $v \in K^2$ is a nonzero vector, i.e., v is a 2-by-1 column vector with entries in K, then $\{v, mv\}$ is a basis of the vector space K^2 over K. So, a matrix representation of the linear transformation defined by m in terms of the basis $\{v, mv\}$ is $\begin{pmatrix} 0 & -\alpha\overline{\alpha} \\ 1 & \alpha+\overline{\alpha} \end{pmatrix}$. The remaining conjugacy class representatives are listed below. Also, by checking orders, one can show these are all of the conjugacy classes.

Conjugacy class representative	Number of conjugacy classes	Number of elements in conjugacy class
$c_4(\alpha) = \begin{pmatrix} 0 & -\alpha\overline{\alpha} \\ 1 & \alpha+\overline{\alpha} \end{pmatrix}$	$\frac{q^2-q}{2}$	q^2-q

Let μ_1 and μ_2 be characters of the multiplicative group K^*. That is, μ_i is a nonzero complex-valued function defined on K^* such that $\mu_i(\alpha\beta) = \mu_i(\alpha)\mu_i(\beta)$ for all $\alpha, \beta \in K^*$. Let B be the set of all upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ in $GL(2, K)$. The function $\mu : B \to \mathbb{C}$ given by

$$\mu \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_1(a)\mu_2(d) \tag{1}$$

defines a 1-dimensional character of B, and each 1-dimensional character of B can be expressed as (1). We also denote μ by (μ_1, μ_2). We will use μ to construct a representation of $GL(2, K)$ in the following way. Let V_μ be the complex vector space given by

$$V_\mu = \{f : GL(2, K) \to \mathbb{C} \mid f(bg) = \mu(b)f(g) \ \forall b \in B\}$$

and define the representation $\hat{\mu} : GL(2, K) \mapsto GL(V_\mu)$ by

$$\hat{\mu}(g)f(x) = f(xg).$$

The representation $\hat{\mu}$ is said to be induced from μ. The dimension of $\hat{\mu}$ is $|G/B| = q+1$. As shown in [1], $\hat{\mu}$ is an irreducible representation iff $\mu_1 \neq \mu_2$. Moreover, the irreducible representations of $GL(2, K)$ induced from (μ_1, μ_2) and (μ_2, μ_1) are equivalent representations. And when $\mu_1 = \mu_2$, one can express $\hat{\mu}$ as a sum of two irreducible representations whose degrees are 1 and q. So, by taking the irreducible components (there are either one or two components) of $\hat{\mu}$ we are able to obtain as many irreducible representations as conjugacy classes of the form $c_1(\alpha)$, $c_2(\alpha)$, and $c_3(\alpha, \beta)$. Thus, there remains more irreducible representations to be found and the number of such representations is the same as the number of conjugacy classes $c_4(\alpha)$. An irreducible representation of $GL(2, K)$ which is not a component of any $\hat{\mu}$ is called a cuspidal representation.

The construction of the cuspidal representations is more involved as shown in [1]. Given a character ν of the multiplicative group of L^*, let $\overline{\nu}$ be the character of L^* defined by $\overline{\nu}(\alpha) = \nu(\overline{\alpha})$ for all $\alpha \in L$. If $\nu \neq \overline{\nu}$, then ν is called a nondecomposable character. We remark that when $\nu = \overline{\nu}$, then there is a character χ of K^* such that $\nu(\alpha) = \chi(\alpha\overline{\alpha})$; so, the choice of the terminology 'nondecomposable' becomes more appropriate. Thus, the number of nondecomposable characters of L^* is $q^2 - q$.

Let L^{*nd} be the set of all nondecomposable characters of L^*. One can associate to each $\nu \in L^{*nd}$ a cuspidal representation ρ_ν in such a way that ρ_ν and $\rho_{\nu'}$ are equivalent representations whenever $\nu = \overline{\nu'}$. This suggests we define an equivalence relation $\sim$ on L^{*nd} by setting $\nu \sim \nu'$ iff $\nu = \nu'$ or $\nu = \overline{\nu'}$. Therefore, there exists a bijection between $L^{*nd}/\sim$, the set of equivalence classes defined by $\sim$, and the cuspidal representations of $GL(2, K)$. As we see, the number of cuspidal representations is $\dfrac{q^2 - q}{2}$ which is the same as the number of conjugacy classes with representative $c_4(\alpha)$.

Let us fix a nonunit character ψ of the additive group K and let ν be a nondecomposable character. To define the corresponding cuspidal representation $\rho = \rho_\nu$, consider the complex vector space $V_\rho = \{f : K^* \to \mathbb{C} \ : \ f \text{ is a complex-valued function}\}$. On B, we define ρ by

$$\left(\rho\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} f\right)(x) = \nu(\delta)\psi(\beta\delta^{-1}x)f(\alpha\delta^{-1}x).$$

Off B, if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\gamma \neq 0$ then ρ is defined by

$$(\rho(g)f)(y) = \sum_{x \in K^*} k(y, x, g)f(x) \tag{2}$$

where

$$k(y, x, g) = \frac{1}{q}\psi\left(\frac{\alpha y + \delta x}{\gamma}\right) \sum_{\substack{u \in L^* \\ u\overline{u} = yx^{-1}det(g)}} \psi\left(-\frac{x}{\gamma}(u + \overline{u})\right)\nu(u).$$

The proof in [1] that ρ is actually a cuspidal representation occupies a good amount of work which we do not discuss here since it is well explained in the cited reference. This outlines a procedure for constructing all the irreducible representations of $GL(2, K)$.

We conclude this article with an example by calculating the nondecomposable characters of L^* where $L = GF(49)$ is the quadratic extension of the field $L = GF(7)$ with seven elements

Example 1 Let $K = \{0, 1, 2, 3, 4, 5, 6\}$ be the field with seven elements. Note, the operations on K are addition and multiplication modulo 7. Since $x^2 + 1$ is an irreducible quadratic polynomial over K, $L = \{a + bx : a, b \in K\}$ is a quadratic extension of K. In the multiplication in L, it is understood that $x^2 = -1 = 6$. Note, $1 + 3x$ is a generator of L^*. To each $q_1 \in \{0, 1, 2, ..., 47\}$, we associate a character ν_{q_1} of L^* by defining

$$\nu_{q_1}((1 + 3x)^n) = exp\left(\frac{2\pi i}{48}q_1 n\right).$$

Here, $exp(z) = e^z$ is the usual exponential function. One finds $\overline{\nu_{q_1}}(1 + 3x) = \nu_{q_1}(\overline{1 + 3x}) = \nu_{q_1}((1 + 3x)^7) = exp\left(\frac{2\pi i}{48}7q_1\right) = \nu_{7q_1}(1 + 3x)$. Thus, $\overline{\nu_{q_1}} = \nu_{7q_1}$. In particular, $\nu_{q_1} = \overline{\nu_{q_1}}$ for $q_1 \in \{0, 8, 16, 24, 32, 40\}$. Therefore, the non-decomposable characters of L^* are given by ν_{q_1} where $q_1 \in ND'$ and

$$ND' = \{0, 1, 2, ..., 47\} - \{0, 8, 16, 24, 32, 40\}. \tag{3}$$

Now, to each $q_1 \in ND'$ one can associate a cuspidal representation $\rho_{(\nu_{q_1})}$. However, $\rho_{(\nu_{q_1})}$ and $\rho_{(\nu_{q_2})}$ are isomorphic representations if $\nu_{q_1} = \overline{\nu_{q_2}}$, i.e., if $q_1 = 7q_2$. Thus, by "taking off multiples of 7 mod 48" in (3), the cuspidal representations $\rho_{\nu_{q_1}}$ are pair-wise non-isomorphic representations whenever $q_1 \in ND$ where

$$ND = \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 20, 25, 26, 27, 33, 34, 41\}.$$

Thus, there are exactly 21 cuspidal representations since $|ND| = 21$, which is exactly the number of conjugacy classes of the form $c_4(\alpha)$.

REFERENCES

1. I. Piatetski-Shapiro, *Complex Representations of GL(2,K) for Finite Fields K*, Contemporary Mathematics, Vol. 16, American Mathematical Society, 1983.