Sums of Powers of Integers

by

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Abstract

The Edwards-Owens-Bloom Algorithm provides a simple technique which generates formulae for the sum of powers of integers. These formulae up to degree 17 were published in the early 17th Century without leaving any records of how they were derived. We present a proof of the Algorithm using simple techniques which where known during that era.

Sums of Powers of Integers

Julien Doucet and A. Saleh-Jahromi

1. Introduction. Sometime ago, we became interested in the sums of powers of integers. One usually sees these formulae in developing the concept of Riemann Sums for the definite integral while teaching Calculus. The following are some of these sums:

$$\sum_{j=1}^{n} 1 = n, \tag{1}$$

$$\sum_{j=1}^{n} j = \tfrac{n(n+1)}{2} = \tfrac{1}{2}n^2 + \tfrac{1}{2}n, \tag{2}$$

$$\sum_{j=1}^{n} j^2 = \tfrac{n(n+1)(2n+1)}{6} = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n, \tag{3}$$

$$\sum_{j=1}^{n} j^3 = \tfrac{n^2(n+1)^2}{4} = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2, \tag{4}$$

$$\sum_{j=1}^{n} j^4 = \tfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n, \tag{5}$$

$$\vdots$$

$$\sum_{j=1}^{n} j^p = \tfrac{1}{p+1}n^{p+1} + \tfrac{1}{2}n^p + (\text{lower powers of } n). \tag{6}$$

These are found in most Calculus books to compute areas. They are usually used to develop the integral formula $\displaystyle\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}$ for integers $n = 0, 1, 2, 3, 4, \ldots, p$. These sums are used in analyzing the number of operations needed to solve linear equations by Gaussian elimination. (See **[**10**]**.)

We became interested in finding a method to derive a formula for the finite sums of powers of integers, $\displaystyle\sum_{j=1}^{n} j^p$ (where p is a non-negative integer), as a polynomial in terms of n. It turns out that we can find a polynomial of degree $p+1$ that will do the job., i.e., for integers $n \geq 0, p \geq 0$,

$$\sum_{j=1}^{n} j^p = P_p(n) \equiv \sum_{j=0}^{p+1} a_j^p n^j \tag{7}$$

or

$$1^p + 2^p + 3^p + \ldots + n^p = P_p(n) \equiv a_{p+1}^p n^{p+1} + a_p^p n^p + a_{p-1}^p n^{p-1} + \ldots + a_2^p n^2 + a_1^p n + a_0^p$$

for some polynomial of n, $P_p(n)$, of degree $p+1$ with (rational) coefficients $a_{p+1}^p, a_p^p, a_{p-1}^p,$

$\ldots, a_2^p, a_1^p, a_0^p$. Methods using finite differences can be used to develop these formulae in general. We will derive formulae for $\sum_{j=1}^{n} j^p$ (for integers $p \geq 0$) expressed as a polynomial of degree $p+1$ using the simplest technique. And we will show how to apply a simple algorithm to generate the coefficients of the polynomials.

One could generate these formulae and then use finite mathematical induction to prove them, case by case. Usually, one verifies the formulae (1)–(6), using finite mathematical induction in a Calculus course. In general, this could be done for the formula (7) after the coefficients are determined. But the question that remains is how to determine the values of the coefficients.

Johann Faulhaber published formulae for (7) up to $p = 17$ in 1631 without leaving any record of how he developed these formulae. See **[6]** p. 119. We give the formulae up to $p = 5$ below.

$$\sum_{j=1}^{n} 1 = n \qquad \sum_{j=1}^{n} j^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$$

$$\sum_{j=1}^{n} j = \frac{1}{2} n^2 + \frac{1}{2} n \qquad \sum_{j=1}^{n} j^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 - \frac{1}{30} n$$

$$\sum_{j=1}^{n} j^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \qquad \sum_{j=1}^{n} j^5 = \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 - \frac{1}{12} n^2$$

Methods using finite differences, factorial polynomials, and summations could generate these formulae. A good presentation of this is in **[8]** p. 22–42. Fortunately, there is a simple algorithm which will generate these formulae. See **[2]**, p. 304–305. Robert W. Owens in **[7]** and David Bloom in **[2]** revealed this algorithm that is very briefly given in **[4]**. We shall refer to that algorithm as the Edwards-Owens-Bloom Algorithm in this paper. As far as we have been able to determine from present day literature on the subject, it is derived using differentiation, Bernoulli numbers, and other methods and techniques which were not known in the early 17th Century. In this article, we shall try to derive the algorithm using simpler methods and techniques which were well understood during Faulhaber's time.

The standard formula for $P_p(n)$ is $P_p(n) = \sum_{j=1}^{p+1} a_j^p n^j$, where $a_j^p = \frac{C_{j-1}^p B_{p+1-j}}{j}$ and $1 \le j \le p+1$ for all $p \ge 0$, where B_k are the Bernoulli numbers. In **[7]**, Owens gives a proof for the algorithm using the Binomial Theorem, mathematical induction, differentiation, and the fact that two polynomials agreeing at infinitely many points must be identical. In **[2]**, Bloom gives a simple proof for this algorithm using Bernoulli numbers.

Inductive arguments was used for arithmetic sequences and sums of integral cubes by Abū Bakr al-Karajī (c. 1000) and continued others (**[5]** p. 255). Levi ben Gerson (1288–1344) used the essentials of the method of induction involving combinatorics and sums involving integers and powers (**[5]** p. 302–304). It appears that induction was well developed by 1653. It seems that Faulhaber would have been familiar with the process when he published his formulae.

The Binomial Theorem was known in Europe as far back as 1527 (**[3]** p. 421). According to Smith, the triangular array called Pascal's Triangle first appeared in print in 1527 in a publication by Apianus. See **[9]** p. 508.

2. Edwards-Owens-Bloom Algorithm. We shall now present some theorems and their proofs and develop the Edwards-Owens-Bloom Algorithm.

Lemma 1. For any constant k and any integer $n \ge 0$, $\sum_{j=1}^{n} k = kn$. In particular, $\sum_{j=1}^{n} 1 = n$.

The proof of the following theorem was suggested to us. It is Pascal's 1654 proof. (See **[1]** p. 202.) It is better and more illuminating than the proof that we had given originally. Students may find it a good exercise to prove it formally by finite induction.

Theorem 1. (i) For any polynomial Q of degree q, there exists a polynomial R such that

$$\sum_{j=1}^{n} Q(j) = R(n) \text{ for } n \ge 0 \text{ and } \deg(R) = q+1.$$

(ii) In particular, for $p, n \ge 0$, there exists a polynomial P_p such that

$$\sum_{j=1}^{n} j^p = P_p(n) \text{ and } \deg\left(P_p\right) = p+1.$$

We shall use the following convention in the material below. Let p, n be integers such that $p, n \geq 0$. From Theorem 1, we have $\sum_{j=1}^{n} j^p = P_p(n)$ for some polynomial P_p of degree $p+1$. So, $P_p(x) = a_{p+1}^p x^{p+1} + a_p^p x^p + a_{p-1}^p x^{p-1} + \ldots + a_2^p x^2 + a_1^p x + a_0^p$ for some constants $a_{p+1}^p, a_p^p, a_{p-1}^p, \ldots, a_2^p, a_1^p, a_0^p$ and $a_{p+1}^p \neq 0$. The coefficients of $P_p(x)$, a_j^p $(0 \leq j \leq p+1)$, must be unique, since $\sum_{j=1}^{n} j^p = P_p(n)$ holds for at least $0 \leq n \leq p+2$.

The results of Theorem 2 below is not new but we shall prove it in a new manner which will require simple algebraic manipulations and finite induction only. We present several lemmata prior to the theorem.

Lemma 2. For any integers $n, p \geq 0$: $$\sum_{j=1}^{n+1} j^p = \sum_{j=0}^{p} (a_j^p + C_j^p) n^j + a_{p+1}^p n^{p+1}. \tag{8}$$

Proof: We have $$\sum_{j=1}^{n} j^p = P_p(n) = \sum_{j=0}^{p+1} a_j^p n^j \text{ for } 0 \leq n, p. \tag{9}$$
Then, it follows from (9) that
$$\begin{aligned}\sum_{j=1}^{n+1} j^p &= \sum_{j=1}^{n} j^p + (n+1)^p = P_p(n) + (1+n)^p = \sum_{j=0}^{p+1} a_j^p n^j + \sum_{j=0}^{p} C_j^p 1^{p-j} n^j \\ &= \sum_{j=0}^{p} a_j^p n^j + a_{p+1}^p n^{p+1} + \sum_{j=0}^{p} C_j^p n^j = \sum_{j=0}^{p} (a_j^p n^j + C_j^p n^j) + a_{p+1}^p n^{p+1} \\ &= \sum_{j=0}^{p} (a_j^p + C_j^p) n^j + a_{p+1}^p n^{p+1}.\end{aligned}$$
□

Lemma 3. For any integers $n, p \geq 0$: $$\sum_{j=1}^{n+1} j^p = \sum_{j=0}^{p} \sum_{k=j}^{p+1} a_k^p C_j^k n^j + a_{p+1}^p n^{p+1}. \tag{10}$$
Proof: Now, it also follows from (9) in Lemma 2 that
$$\begin{aligned}\sum_{j=1}^{n+1} j^p &= P_p(n+1) = \sum_{j=0}^{p+1} a_j^p (n+1)^j = \sum_{j=0}^{p+1} a_j^p (1+n)^j = \sum_{j=0}^{p+1} \left(a_j^p \sum_{k=0}^{j} C_k^j 1^{j-k} n^k \right) \\ &= \sum_{j=0}^{p+1} \sum_{k=0}^{j} a_j^p C_k^j n^k = \sum_{k=0}^{p+1} \sum_{j=k}^{p+1} a_j^p C_k^j n^k = \sum_{j=0}^{p+1} \sum_{k=j}^{p+1} a_k^p C_j^k n^j \\ &= \sum_{j=0}^{p} \sum_{k=j}^{p+1} a_k^p C_j^k n^j + a_{p+1}^p C_{p+1}^{p+1} n^{p+1} = \sum_{j=0}^{p} \sum_{k=j}^{p+1} a_k^p C_j^k n^j + a_{p+1}^p n^{p+1}.\end{aligned}$$
□

Lemma 4. For integers j, p such that $1 \leq j \leq p+1$: $$a_j^p = \frac{1}{j} \left[C_{j-1}^p - \sum_{k=j+1}^{p+1} C_{j-1}^k a_k^p \right]. \tag{11}$$

Proof: Combining Lemmata 2 and 3, i.e., (8) and (10), we get:

$\sum_{j=0}^{p}(a_j^p + C_j^p)n^j + a_{p+1}^p n^{p+1} = \sum_{j=0}^{p}\sum_{k=j}^{p+1} a_k^p C_j^k n^j + a_{p+1}^p n^{p+1}$ for $0 \le n, p$.

By equating coefficients of like terms of n^j, we get for $0 \le j \le p$:

$a_j^p + C_j^p = \sum_{k=j}^{p+1} a_k^p C_j^k = a_j^p \cdot 1 + a_{j+1}^p \cdot (j+1) + \sum_{k=j+2}^{p+1} a_k^p C_j^k = a_j^p + (j+1)a_{j+1}^p + \sum_{k=j+2}^{p+1} a_k^p C_j^k.$

Then, $C_j^p = (j+1)a_{j+1}^p + \sum_{k=j+2}^{p+1} a_k^p C_j^k$ and $(j+1)a_{j+1}^p = C_j^p - \sum_{k=j+2}^{p+1} a_k^p C_j^k$. Dividing by $j+1$, we get $a_{j+1}^p = \frac{1}{j+1}\left[C_j^p - \sum_{k=j+2}^{p+1} a_k^p C_j^k\right]$ for $0 \le j \le p$ or $1 \le j+1 \le p+1$. Replacing $j+1$ with j, we get (11). □

Lemma 5. For integers k and p such that $0 \le k \le p$: $a_{p+2-k}^{p+1} = \frac{p+1}{p+2-k} a_{p+1-k}^p$. (12)

Proof: We will prove Lemma 5, i.e., (12) by strong induction on k.

Let $0 \le p$. Then $1 \le p+1$. From Lemma 4, i.e., (11), we have:

$a_{p+1}^p = \frac{1}{p+1}\left[C_p^p - \sum_{k=p+2}^{p+1} C_p^k a_k^p\right] = \frac{1}{p+1}[1-0] = \frac{1}{p+1} \cdot 1 = \frac{1}{p+1}$. Therefore,

$a_{p+2}^{p+1} = \frac{1}{p+2} = \frac{p+1}{p+2} \cdot \frac{1}{p+1} = \frac{p+1}{p+2} \cdot a_{p+1}^p$. Hence, (12) holds for $0 = k \le p$.

Suppose (12) holds for all k such that $0 \le k \le K$. It suffices to show that (12) then holds for $k = K+1$. Let $K+1 \le p$. Then $0 \le k \le K < K+1 \le p$. Hence,

$$0 \le K+1 \le p. \tag{13}$$

Also, we now have $1 \le 2 = K+1+1-K \le p+1-K \le p+1 \le p+2$, i.e.,

$$1 \le p+1-K \le p+2. \tag{14}$$

Since $K+1 \le p$, then $1 \le p-K$. Since $0 \le K$, then $p-K \le p \le p+1$. Combining these, we have

$$1 \le p-K \le p+1. \tag{15}$$

And, it follows that

$$\begin{aligned} a_{p+2-(K+1)}^{p+1} = a_{p+1-K}^{p+1} &= \frac{1}{p+1-K}\left[C_{p-K}^{p+1} - \sum_{\ell=p+2-K}^{p+2} C_{p-K}^{\ell} a_\ell^{p+1}\right] \quad \text{by (11) and (14)} \\ &= \frac{1}{p+1-K}\left[\frac{(p+1)!}{(K+1)!\,(p-K)!} - \sum_{\ell=0}^{K} C_{p-K}^{p+2-\ell} a_{p+2-\ell}^{p+1}\right] \\ &= \frac{1}{p+1-K}\left[\frac{(p+1)!}{(K+1)!\,(p-K)!} - \sum_{\ell=0}^{K} \frac{(p+2-\ell)!}{(K-\ell+2)!\,(p-K)!} a_{p+2-\ell}^{p+1}\right] \end{aligned}$$

$$= \tfrac{1}{p+1-K}\left[\tfrac{(p+1)!}{(K+1)!\,(p-K)!} - \sum_{\ell=0}^{K} \tfrac{(p+2-\ell)!}{(K-\ell+2)!\,(p-K)!}\;\tfrac{(p+1)}{(p+2-\ell)}\;a^p_{p+1-\ell}\right]\begin{cases}\text{since (12) holds}\\ \text{for } 0 \le \ell \le K\end{cases}$$

$$= \tfrac{1}{p+1-K}\left[\tfrac{(p+1)!}{(K+1)!\,(p-K)!} - \sum_{\ell=p-K+1}^{p+1} \tfrac{(1+\ell)!}{(K+\ell-p+1)!\,(p-K)!}\;\tfrac{(p+1)}{(1+\ell)}\;a^p_{\ell}\right]$$

$$= \tfrac{1}{p+1-K}\left[\tfrac{(p+1)}{(p-K)}\;\tfrac{p!}{(K+1)!\,(p-K-1)!} - \tfrac{(p+1)}{(p-K)}\sum_{\ell=p-K+1}^{p+1} \tfrac{\ell!}{(K+\ell-p+1)!\,(p-K-1)!}\;a^p_{\ell}\right]$$

$$= \tfrac{(p+1)}{(p+1-K)}\,\tfrac{1}{(p-K)}\left[C^p_{p-K-1} - \sum_{\ell=p-K+1}^{p+1} C^{\ell}_{p-K-1}a^p_{\ell}\right]$$

$$= \tfrac{(p+1)}{(p+1-K)}\;a^p_{p-K} \qquad \text{by (11) and (15)}$$

$$= \tfrac{p+1}{p+2-(K+1)}\;a^p_{p+1-(K+1)} \text{ for } 0 \le K+1 \le p \qquad \text{by (13).}$$

Hence, (12) is true for $k = K+1$. □

Theorem 2. If $0 \le p$ and $1 \le j \le p+1$, then $a^{p+1}_{j+1} = \frac{p+1}{j+1}a^p_j$.

Proof: Letting $j = p+1-k$ in Lemma 5, i.e., (12), we get:

$$a^{p+1}_{j+1} = a^{p+1}_{p+2-k} = \tfrac{p+1}{p+2-k}a^p_{p+1-k} = \tfrac{p+1}{j+1}a^p_j \text{ for } 0 \le k \le p.$$

But, $0 \le k \le p \iff -p \le -k \le 0 \iff -p+p+1 \le -k+p+1 \le 0+p+1$ $\iff 1 \le p+1-k \le p+1 \iff 1 \le j \le p+1$. Hence, $a^{p+1}_{j+1} = \frac{p+1}{j+1}a^p_j$ for $1 \le j \le p+1$. □

The following theorems and corollary do not yield new results. Their proofs are short and require little effort with the above results.

Theorem 3. For any $p \ge 0$, $a^p_0 = 0$.

Proof: We have $\sum_{j=1}^{n} j^p = P_p(n) = \sum_{j=0}^{p+1} a^p_j n^j$ for $0 \le n, p$. Setting $n = 0$, we get $a^p_0 = 0$. □

Theorem 4. For any $p \ge 0$, $\sum_{j=1}^{p+1} a^p_j = 1$.

Proof: By Theorem 3, $\sum_{j=1}^{n} j^p = \sum_{j=1}^{p+1} a^p_j n^j$ for $0 \le n, p$. Setting $n = 1$, we get $\sum_{j=1}^{p+1} a^p_j = 1$. □

Corollary 1. For any $p \ge 0$, $a^p_1 = 1 - \sum_{j=2}^{p+1} a^p_j$.

Using Theorem 4, we get $P_p(n) = \sum_{j=0}^{p+1} a_j^p n^j = \sum_{j=1}^{p+1} a_j^p n^j + a_0^p = \sum_{j=1}^{p+1} a_j^p n^j + 0 = \sum_{j=1}^{p+1} a_j^p n^j$ for $0 \leq p$. Hence, $\sum_{j=1}^{n} j^p = P_p(n) = \sum_{j=1}^{p+1} a_j^p n^j = a_{p+1}^p x^{p+1} + a_p^p x^p + a_{p-1}^p x^{p-1} + \ldots + a_3^p x^3 + a_2^p x^2 + a_1^p x$ for $0 \leq n, p$. Using Theorem 2 and Corollary 1, we can develop the Edwards-Owens-Bloom Algorithm. The Edwards-Owens-Bloom Algorithm generates the coefficients of $P_{p+1}(n)$ from the coefficients of $P_p(n)$ for $p \geq 0$. The algorithm is as follows:

The Edwards-Owens-Bloom Algorithm

(1) Given: $P_p(n) = \sum_{j=1}^{p+1} a_j^p n^j$ with known coefficients a_j^p $(1 \leq j \leq p+1)$.

(2) Multiply each a_j^p by $\frac{p+1}{j+1}$ to produce a_{j+1}^{p+1} $(1 \leq j \leq p+1)$. This will generate the coefficients of $P_{p+1}(n)$: a_j^{p+1} for $2 \leq j \leq p+2$. All of the coefficients of $P_{p+1}(n)$ will have been determined, except for a_1^{p+1}.

(3) To produce the last (missing) coefficient of $P_{p+1}(n)$, a_1^{p+1}:

(a) add all of the coefficients of $P_{p+1}(n)$ generated in Step (2);

(b) subtract the sum in (a) from 1; and

(c) a_1^{p+1} is the resultant difference produced in (b).

The above algorithm is illustrated in the diagram below.

$$\begin{array}{cccccccccccc}
a^p_{p+1}x^{p+1} + & a^p_p x^p & + a^p_{p-1}x^{p-1} + \ldots + & a^p_j x^j & + \ldots + & a^p_3 x^3 & + & a^p_2 x^2 & + & a^p_1 x \\
\big\downarrow \times \frac{p+1}{p+2} & \big\downarrow \times \frac{p+1}{p+1} & \big\downarrow \times \frac{p+1}{p} \quad \cdots & \big\downarrow \times \frac{p+1}{j+1} & \cdots & \big\downarrow \times \frac{p+1}{4} & & \big\downarrow \times \frac{p+1}{3} & & \big\downarrow \times \frac{p+1}{2} \\
a^{p+1}_{p+2}x^{p+2} + a^{p+1}_{p+1}x^{p+1} + & a^{p+1}_p x^p & + \ldots + & a^{p+1}_{j+1}x^{j+1} & + \ldots + & a^{p+1}_4 x^4 + & a^{p+1}_3 x^3 & + & a^{p+1}_2 x^2 & + a^{p+1}_1 x \\
 & & & & & & & & & \| \\
 & & & & & & & & & 1 - \sum_{j=2}^{p+2} a_j^{p+1}
\end{array}$$

3. Epilogue. In **[7]**, Owens used the binomial theorem, mathematical induction, and differentiation to derive the Edwards-Owens-Bloom Algorithm. In **[2]**, Bloom used Bernoulli numbers to justify the algorithm. In our opinion, Faulhaber could not have derived the algorithm as did Owens in **[7]** and as did Bloom in **[2]**. In this article, the formulae for (7) could be derived by using finite differences. This method can be continued to derive the formula in (7) for any arbitrarily fixed integer $p \geq 0$. The Edwards-Owens-Bloom Algorithm was derived in this article using simple methods, i.e., using algebraic manipulations which were well understood in the 17th Century. Perhaps, he used method of finite differences or, as we think more likely, he developed the Edwards-Owens-Bloom Algorithm which could have been derived using the algebraic manipulations in Lemma 1, Theorem 1, Lemmata 2–5, Theorems 2–4, and Corollary 1.

Acknowledgments. We wish to thank George Boros, Carlos G. Spaht II, Dan Kalman, and Carl Libis for their encouragement to further investigate this topic and to write an article on it. George Boros was particularly helpful to us with some of the reference materials. Carl Libis pointed out some minor errors in this article and expressed some of his opinions about the material and history involved. We are grateful to him for his help and suggestions. Also, we wish to express our appreciation for suggestions for some improvements from an anonymous person.

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